$g(x) = x$. Thus $g: B^n \longrightarrow B^n$ is the identity on ∂B^n. We only need show that g is smooth to obtain a contradiction to the retraction theorem and thereby complete the proof. Since x is in the line segment between $f(x)$ and $g(x)$, we may write the vector $g(x) - f(x)$ as a multiple t times the vector $x - f(x)$, where $t \geq 1$. Thus $g(x) = tx + (1 - t)f(x)$. If t depends smoothly on x, then g is smooth. Take the dot product of both sides of this formula. Because $|g(x)| = 1$, you obtain the formula

$$t^2 |x - f(x)|^2 + 2tf(x) \cdot [x - f(x)] + |f(x)|^2 - 1 = 0.$$

The latter may look ugly, but it has the redeeming virtue of being a quadratic polynomial with a unique positive root. (There is also a root with $t \leq 0$, corresponding to the point where the line from x through $f(x)$ hits the boundary.) Now we need only substitute into the quadratic formula of high school to obtain an expression for t in terms of smooth functions of x. Q.E.D.

EXERCISES

1. Any one-dimensional, compact, connected submanifold of $\mathbf{R}^3$ is diffeomorphic to a circle. But can it be deformed into a circle within $\mathbf{R}^3$? Draw some pictures, or try with string.

2. Show that the fixed point in the Brouwer theorem need not be an interior point.

3. Find maps of the solid torus into itself having no fixed points. Where does the proof of the Brouwer theorem fail?

4. Prove that the Brouwer theorem is false for the open ball $|x|^2 < a$. [HINT: See Chapter 1, Section 1, Exercise 4.]

5. Prove the Brouwer theorem for maps of $[0, 1]$ directly, without using regular values.

6. Prove the Brouwer theorem for continuous maps $f: B^n \longrightarrow B^n$. Use the Weierstrass Approximation Theorem, which says that for any $\epsilon > 0$ there exists a polynomial mapping $p: \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that $|f - p| < \epsilon$ on B^n. (One reference: Rudin's *Principles of Mathematical Analysis*, "The Stone-Weierstrass Theorem.") [HINT: First show that given $\delta > 0$, you can find p so that $|f - p| < \delta$ and $p: B^n \longrightarrow B^n$. Now use the fact that if f has no fixed points, $|f(x) - x| > c > 0$ on B^n.]

7. As a surprisingly concrete application of Brouwer, prove this theorem of Frobenius: if the entries in an $n \times n$ real matrix A are all nonnegative,

then A has a real nonnegative eigenvalue. [HINT: It suffices to assume A nonsingular; otherwise 0 is an eigenvalue. Let A also denote the associated linear map of $\mathbf{R}^n$, and consider the map $v \longrightarrow Av/|Av|$ restricted to $S^{n-1} \longrightarrow S^{n-1}$. Show that this maps the "first quadrant"

$$Q = \{(x_1, \ldots, x_n) \in S^{n-1} : \text{all } x_i \geq 0\}$$

into itself. It is not hard to prove, although we don't expect you to bother, that Q is homeomorphic with B^{n-1}; that is, there exists a continuous bijection $Q \longrightarrow B^{n-1}$ having a continuous inverse. Now use Exercise 6.] See Figure 2-8.

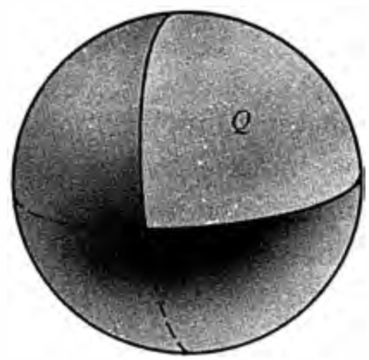
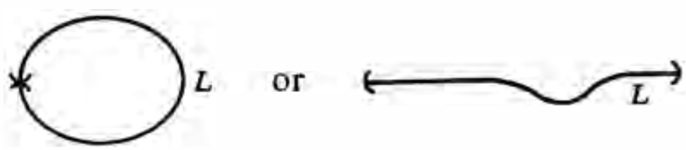


Figure 2-8

8. Suppose that dim $X = 1$ and L is a subset diffeomorphic to an open interval in $\mathbf{R}^1$. Prove that $\bar{L} - L$ consists of at most two points:



This is needed for the classification of one-manifolds. [HINT: Given $g: (a, b) \xrightarrow{\approx} L$, let $p \in \bar{L} - L$. Let J be a closed subset of X diffeomorphic to $[0, 1]$, such that 1 corresponds to p and 0 corresponds to some $g(t) \in L$. Prove that J contains either $g(a, t)$ or $g(t, b)$, by showing that the set $\{s \in (a, t) : g(s) \in J\}$ is open and closed in (a, b).]

§3 Transversality

Earlier we proved that transversality is a property that is stable under small perturbations, at least for maps with compact domains. From Sard's theorem we shall deduce the much more subtle and valuable fact that transversality is a generic quality: any smooth map $f: X \longrightarrow Y$, no matter how bizarre its behavior with respect to a given submanifold Z in Y, may be de-